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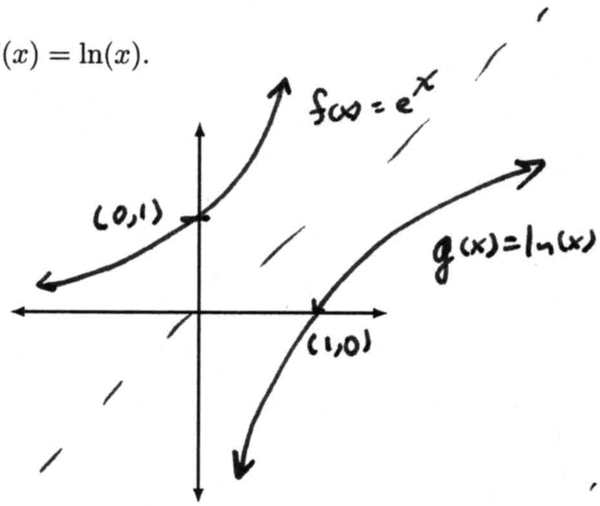
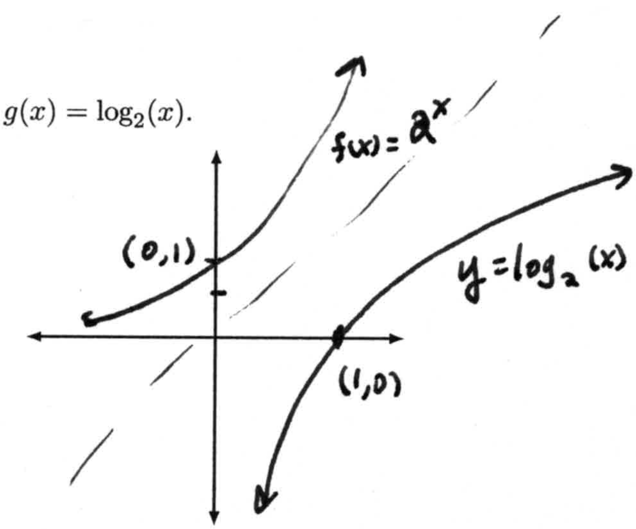
Exponential and Logarithmic Functions

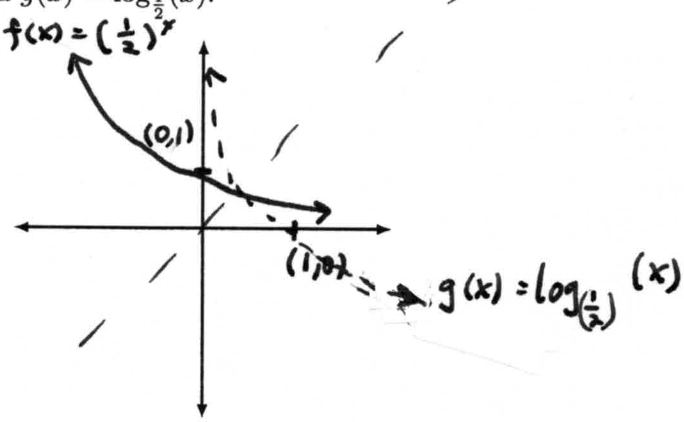
1. Graph $f(x) = e^x$ and $f(x) = \ln(x)$.



2. Sketch $f(x) = 2^x$ and $g(x) = \log_2(x)$.



3. Sketch $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{\frac{1}{2}}(x)$.



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4. Rewrite the following functions in the form $a \cdot b^x$. Then, use a calculator to compute b.

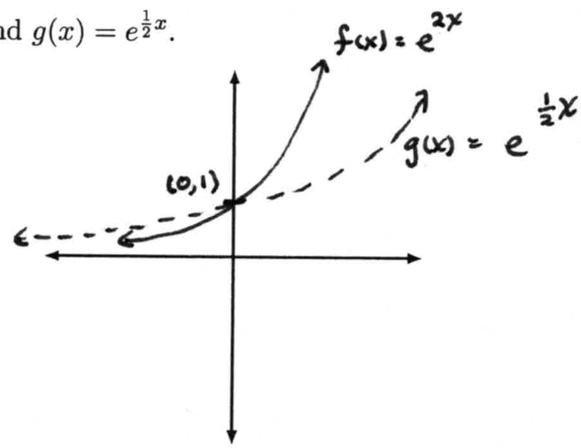
(a) $f(x) = e^{2x}$ $= 1 \cdot (e^2)^x = 1 \cdot (7.389\ldots)^x$

(b) $g(x) = e^{\frac{1}{2}x}$ $= 1 \cdot (e^{\frac{1}{2}})^x = 1 \cdot (1.649\ldots)^x$

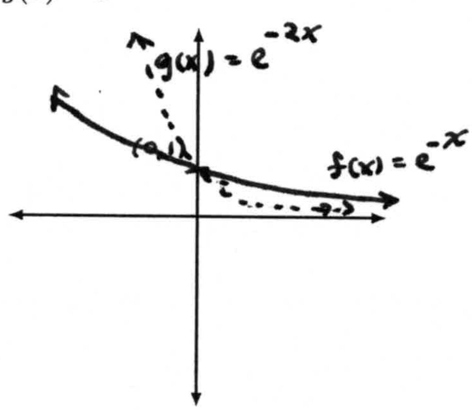
(c) $f(x) = e^{-x}$ $= 1 \cdot \left(\frac{1}{e}\right)^x = 1 \cdot (0.368\ldots)^x$

(d) $g(x) = e^{-2x}$ $= 1 \cdot \left(\frac{1}{e^2}\right)^x = 1 \cdot (0.1353\ldots)^x$

5. Sketch $f(x) = e^{2x}$ and $g(x) = e^{\frac{1}{2}x}$.



6. Sketch $f(x) = e^{-x}$ and $g(x) = e^{-2x}$



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7. Evaluate the following expressions:

   (a) $\log_3(9)$ $= \log_3(3^2) = 2$

   (b) $\log_2\left(\sqrt[3]{2}\right)$ $= \log_2(2^{1/3}) = \frac{1}{3}$

   (c) $\log_3\left(\frac{1}{\sqrt{3}}\right)$ $= \log_3(3^{-1/2}) = -1/2$

   (d) $\ln\left(\frac{1}{e}\right)$ $= \ln(e^{-1}) = -1$

   (e) $\log_2(16)$ $= \log_2(2^4) = 4$

   (f) $\log_4(16)$ $= \log_4(4^2) = 2$

   (g) $\log_{16}(16)$ $= \log_{16}(16^1) = 1$

   (h) $\ln(e) = \ln(e^1) = 1$

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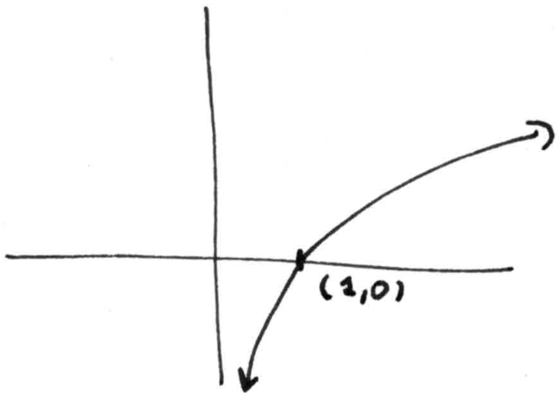
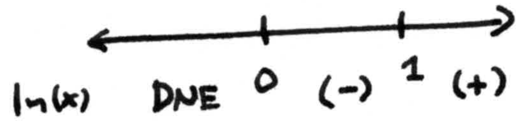
8. Sketch $y = \ln(x)$ and give its sign chart. For what x is $\ln(x) > 0$? For what x is $\ln(x) < 0$?



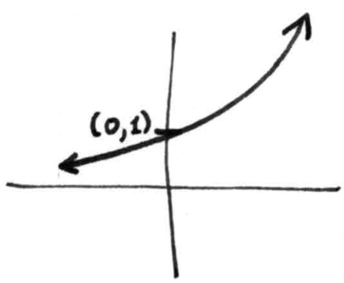


$\ln(x) < 0$ on $(0, 1)$

and

$\ln(x) > 0$ on $(1, \infty)$

9. Sketch $y = e^x$. For what x is $e^x > 1$? For what x is $e^x < 1$?



$e^x > 1$ when x is in $(0, \infty)$

$e^x < 1$ when x is in $(-\infty, 0)$

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10. Simplify the following expressions

(a) $e^{5\ln(2)}$

$$= \left(e^{\ln(2)}\right)^5 =$$

$$= 2^5$$

(b) $\ln(e^{5x})$

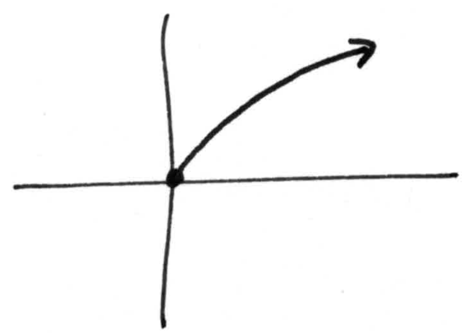
$$= 5x$$

11. Simplify and sketch the following functions $f(x) = e^{\frac{1}{2}\ln(x)}$ and $g(x) = e^{x\ln(\frac{1}{2})}$.

$$f(x) = e^{\frac{1}{2}\cdot\ln(x)} = \left(e^{\ln(x)}\right)^{\frac{1}{2}}$$

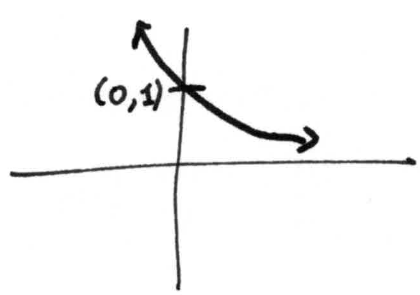
$$= x^{1/2}$$

$$= \sqrt{x}$$



$$g(x) = e^{x\cdot\ln(\frac{1}{2})} = \left(e^{\ln(\frac{1}{2})}\right)^x$$

$$= \left(\frac{1}{2}\right)^x$$



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14. Rewrite the following function in the form $f(t) = a \cdot b^t$

$$f(t) = 10e^{\left(\frac{\ln(2)t}{3}\right)}$$

$$10 \cdot \left(e^{\ln(2)}\right)^{\frac{t}{3}} = 10 \cdot \left((2)^{\frac{1}{3}}\right)^{t} = 10 \cdot \left(\sqrt[3]{2}\right)^{t}$$

15. Rewrite the following function in the form $f(x) = a \cdot b^x$. **Simplify completely.**

$$f(x) = 10 \cdot 2^{3x-1}$$

$$= 10 \cdot 2^{3x} \cdot 2^{-1}$$

$$= 10 \cdot \left(2^{3}\right)^{x} \cdot \frac{1}{2}$$

$$= 5 \cdot 8^{x}$$

16. Rewrite the following function in the form $f(x) = a \cdot e^{kx}$

$$f(x) = 3 \cdot 4^x$$

$$= 3 \cdot \left(e^{\ln(4)}\right)^{x}$$

$$= 3 \cdot e^{\ln(4) \cdot x}$$

17. Rewrite the following function in the form $f(x) = a \cdot e^{kx}$

$$f(x) = 40 \cdot (2.19)^{2x}$$

$$= 40 \cdot \left(e^{\ln(2.19)}\right)^{2x}$$

$$\boxed{= 40 \cdot e^{2 \cdot \ln(2.19) \cdot x}}$$

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18. Rewrite the following expression as a single logarithm:

$$\ln(x+5) - \ln(x-2) + \ln(x)$$

$$= \ln\left(\frac{x+5}{x-2}\right) + \ln(x)$$

$$= \ln\left(\frac{x(x+5)}{(x-2)}\right)$$

19. Rewrite the following expression as a single logarithm:

$$2\ln(x+1) + 3\ln(x+2)$$

$$= \ln((x+1)^2) + \ln((x+2)^3)$$

$$= \ln\left((x+1)^2(x+2)^3\right)$$

20. Rewrite the following expression as a single logarithm:

$$\ln(x+3) - 2\ln(x) + \ln(x+1)$$

$$= \ln(x+3) - \ln(x^2) + \ln(x+1)$$

$$= \ln\left(\frac{(x+3)(x-1)}{x^2}\right)$$

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21. Use the laws of logarithms to rewrite the following expression as sums and/or differences of logarithmic expressions that do not contain logarithms of products, quotients, or powers.

$$\ln\left(\frac{x^4y^6}{z^3}\right)$$

$$= \ln(x^4) + \ln(y^6) - \ln(z^3)$$

$$= 4\cdot\ln(x) + 6\cdot\ln(y) - 3\cdot\ln(z)$$

22. Use the laws of logarithms to rewrite the following expression as sums and/or differences of logarithmic expressions that do not contain logarithms of products, quotients, or powers.

$$\ln\left(\frac{x^{24}\sqrt{y+1}}{z^3}\right)$$

$$= \ln(x^{24}) + \ln\left((y+1)^{1/2}\right) - \ln(z^3)$$

$$= 24\cdot\ln(x) + \frac{1}{2}\cdot\ln(y+1) - 3\cdot\ln(z)$$

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Equations of Exponentials and Logarithms

1. Solve the following equation for x

$$400e^{7x} = 300$$

$$e^{7x} = \frac{300}{400} = \frac{3}{4}$$

$$\ln(e^{7x}) = \ln\left(\frac{3}{4}\right)$$

$$7x = \ln\left(\frac{3}{4}\right)$$

$$\boxed{x = \frac{\ln\left(\frac{3}{4}\right)}{7}}$$

2. Find the values for x which satisfy the following equation

$$30e^{4x+1} - 5 = 85$$

① get exp term alone

$$30 \cdot e^{4x+1} = 90$$

$$e^{4x+1} = \frac{90}{30} = 3$$

② take ln of both sides

$$\ln(e^{4x+1}) = \ln(3)$$

$$4x+1 = \ln(3)$$

③ solve for x

$$4x = \ln(3) - 1$$

$$\boxed{x = \frac{\ln(3)-1}{4}}$$

3. Find the values for x which satisfy the following equation

$$30e^{4x} = 9e^{1-x}$$

① exp terms are alone.

② take ln of both sides & break apart

$$\ln(30 \cdot e^{4x}) = \ln(9 \cdot e^{1-x})$$

$$\ln(30) + \ln(e^{4x}) = \ln(9) + \ln(e^{1-x})$$

$$\ln(30) + 4x = \ln(9) + 1 - x$$

③ solve for x

$$5x = \ln(9) - \ln(30) + 1$$

$$\boxed{x = \frac{\ln(9)-\ln(30)+1}{5}}$$

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4. Solve the following equation for x

$$15^x + 10 = 45$$

$$15^x = 35$$

$$\ln(15^x) = \ln(35)$$

$$x \cdot \ln(15) = \ln(35)$$

$$\boxed{x = \frac{\ln(35)}{\ln(15)}}$$

5. Solve the following equation for x

$$5^{x+1} = 7^{2x-1}$$

① take ln of both sides & break apart

$$\ln(5^{x+1}) = \ln(7^{2x-1})$$

$$(x+1) \cdot \ln(5) = (2x-1) \cdot \ln(7)$$

$$x \cdot \ln(5) + \ln(5) = 2x \cdot \ln(7) - \ln(7)$$

② collect x - terms

$$\ln(5) + \ln(7) = 2x \cdot \ln(7) - x \cdot \ln(5)$$

③ factor out x & solve

$$\ln(5) + \ln(7) = x\left(2 \cdot \ln(7) - \ln(5)\right)$$

$$\boxed{x = \frac{\ln(5) + \ln(7)}{2 \cdot \ln(7) - \ln(5)}}$$

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Exponential Growth and Decay

1. What is the basic formula for exponential growth and decay?
   What is the physical meaning of each constant?

$$P(t) = P_0 \cdot e^{kt}$$

P_0 = initial population

k = growth constant
= rate of growth/person

also written

$$f(t) = a \cdot e^{kt}$$

$a = f(0)$
k = growth constant.

$\boxed{\text{k is POSITIVE in exp growth \& NEGATIVE in exp decay}}$

2. In exponential decay, what does the term "half life" mean?

the half life is the
time you must wait
to get the quantity cut in half

Eg: If you start w/ 10 grams & half life is 7 years,
$P(7) = 5$
and $P(14) = 2.5$

3. In exponential growth, what does the term "doubling time" mean?

the doubling time is the
time you must wait
to get the quantity to double

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4. Suppose that a population of bacteria is growing in a petri dish. Suppose also that the first time you look at the dish you count 20 bacteria, and that you count 200 bacteria in the dish 3 hours later.

   (a) Find a formula for the population as a function of the number of hours t since your first measurement.

   (b) Estimate the number of bacteria after 24 hours.

   (c) How long until the dish contains 1000 bacteria?

   (d) Find the number of hours needed for the population to double.

(a) $P(t) = P_0 \cdot e^{kt}$

$P_0 = 20$

$P(t) = 20 \cdot e^{kt}$ ← need to find k

know $P(3) = 200 = 20 \cdot e^{k \cdot 3}$

$200 = 20\, e^{k \cdot 3}$

$10 = e^{k \cdot 3}$

$\ln(10) = k \cdot 3$

$k = \dfrac{\ln(10)}{3}$

So $\boxed{P(t) = 20 \cdot e^{\left(\frac{\ln(10)}{3}\right)t}}$

(b) $P(24) = 20 \cdot e^{\left(\frac{\ln(10)}{3}\right)24} = 20 \cdot e^{\ln(10) \cdot \frac{24}{3}} = 20 \cdot \left(e^{\ln(10)}\right)^{\frac{24}{3}}$

$= 20 \cdot (10)^{\frac{24}{3}}$

(c) $1000 = P(t)$

$\Leftrightarrow$

$1000 = 20 \cdot e^{\left(\frac{\ln(10)}{3}t\right)}$

$50 = e^{\frac{\ln(10)}{3}t}$

$\ln(50) = \frac{\ln(10)}{3}t$

$t = \dfrac{3 \cdot \ln(50)}{\ln(10)}$

(d) $40 = P(t)$

$\Leftrightarrow$

$40 = 20 \cdot e^{\frac{\ln(10)}{3}t}$

$\Leftrightarrow$

$2 = e^{\left(\frac{\ln(10)}{3}t\right)}$

$\ln(2) = \frac{\ln(10)}{3}t$

$t = \dfrac{3 \cdot \ln(2)}{\ln(10)}$

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5. Suppose that you begin with 100 grams of a radioactive substance. Suppose also that the substance has a half life of 3 years.

   (a) Find a formula for the amount of radioactive substance remaining after t years.

   (b) What is the weight of the radioactive substance that remains after 9 years?

   ~~(c) How long must you wait for the weight of radioactive substance to drop to 10 grams?~~

(a) $P(t) = P_0 \cdot e^{kt}$

$P_0 = 100$ g.

$P(t) = 100 \cdot e^{kt}$

need to find k.

know

$P(3) = 50 = 100 \cdot e^{k \cdot 3}$

Solve for k: $50 = 100 \cdot e^{3k}$

$\frac{1}{2} = e^{3k}$

$\ln(\frac{1}{2}) = 3k$

$k = \frac{\ln(\frac{1}{2})}{3}$

$$\boxed{P(t) = 100\, e^{\frac{\ln(\frac{1}{2})}{3} t}}$$

(b) $P(\boxed{9}) = 100 \cdot e^{\frac{\ln(\frac{1}{2})}{3} \boxed{9}}$

$= 100\, e^{\ln(\frac{1}{2}) \cdot 3}$

$= 100 \cdot \left(e^{\ln(\frac{1}{2})}\right)^3$

$= 100 \cdot \left(\frac{1}{2}\right)^3$

$P(9) = \frac{100}{8}$

(c) find t s.t.

$P(t) = 10$.

$10 = 100 \cdot e^{\frac{\ln(\frac{1}{2})}{3} t}$

$\frac{1}{10} = e^{\frac{\ln(\frac{1}{2})}{3} t}$

$\ln(\frac{1}{10}) = \frac{\ln(\frac{1}{2})}{3} t$

$\frac{3 \cdot \ln(\frac{1}{10})}{\ln(\frac{1}{2})} = t$

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Additional Exponential Models

1. When you take a 18lb turkey out of a 40° refrigerator and place it in a 350° oven, its temperature after x hours is modeled[1] by

$$T(x) = 350 + (40 - 350) \cdot e^{-0.15x}$$

(a) How long must you wait until the internal temperature is 165°?[2]

want x so that

$$165 = 350 + (-310) \cdot e^{-0.15x}$$

$$-180 = -310 \cdot e^{-0.15x}$$

$$\frac{180}{310} = \frac{-180}{-310} = e^{-0.15x}$$

$$\ln\left(\frac{180}{310}\right) = -0.15x \Rightarrow x = \frac{\ln\left(\frac{180}{310}\right)}{-0.15} \left(\approx 3.6 \text{ hours}\right)$$

(b) Find a formula for $T^{-1}(x)$. Use this formula to compute the time required for the Turkey's internal temperature to reach 100°, 200°, and 300°.

① set up formula

$$y = 350 - 310 \cdot e^{-0.15x}$$

② solve for the input x giving that output

$$y - 350 = -310 \cdot e^{-0.15x}$$

$$\frac{y-350}{-310} = e^{-0.15x}$$

$$\ln\left(\frac{y-350}{-310}\right) = -0.15x$$

$$T^{-1}(y) = x = \frac{\ln\left(\frac{y-350}{-310}\right)}{-0.15}$$

③ Change var's

$$\boxed{T^{-1}(x) = \frac{\ln\left(\frac{x-350}{-310}\right)}{-0.15}}$$

$$T^{-1}(100) = \frac{\ln\left(\frac{-250}{-310}\right)}{-0.15} \approx 1.4 \text{ hrs}$$

$$T^{-1}(200) = \frac{\ln\left(\frac{-150}{-310}\right)}{-0.15} \approx 4.8 \text{ hrs}$$

$$T^{-1}(300) = \frac{\ln\left(\frac{-50}{-310}\right)}{-0.15} \approx 12.1 \text{ hrs}$$

[1] k was computed using the cook times at http://allrecipes.com/howto/turkey-cooking-time-guide/

[2] the temperature of a fully cooked Turkey

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2. The spread of a zombie outbreak in a neighborhood of 100 people is modeled by the function

$$P(x) = \frac{100}{1 + 99 \cdot e^{-0.5x}}$$

where x is measured in days.

(a) How many zombies are there on day 0 of the outbreak? What about after 10 days?

$$P(0) = \frac{100}{1+99\cdot e^{0}} = \frac{100}{1+99} = 1 \text{ zombie}$$

$$P(10) = \frac{100}{1+99\cdot e^{-5}} \approx 60 \text{ zombies}$$

(b) How long do you have until one quarter of the neighborhood (25 people) are zombies.

find x so that

$$25 = \frac{100}{1+99\cdot e^{-0.5x}}$$

$$25(1+99\cdot e^{-0.5x}) = 100$$

$$25 + (25)(99)e^{-0.5x} = 100$$

$$(25)(99)\,e^{-0.5x} = 75$$

$$e^{-0.5x} = \frac{75}{(25)(99)} \quad \left(= \frac{3}{99} = \frac{1}{11}\right)$$

$$-0.5x = \ln\left(\frac{75}{(25)(99)}\right) \quad \left(= \ln\left(\frac{1}{11}\right)\right)$$

$$\boxed{x = \frac{\ln\left(\frac{(75)}{(25)(99)}\right)}{-0.5} = \frac{\ln\left(\frac{1}{11}\right)}{-0.5}} \approx 4.8 \text{ days}$$

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The spread of a zombie outbreak in a neighborhood of 100 people is modeled by the function

$$P(x) = \frac{100}{1 + 99 \cdot e^{-0.5x}}$$

(c) Find a formula for $P^{-1}(x)$.

① Set up eqn

$$y = \frac{100}{1 + 99 \cdot e^{-0.5x}}$$

② solve for the x that outputs this y.

$$y(1 + 99 \cdot e^{-0.5x}) = 100$$

$$y + 99 \cdot y \cdot e^{-0.5x} = 100$$

$$99 \cdot y \cdot e^{-0.5x} = 100 - y$$

$$e^{-0.5x} = \frac{100-y}{99y}$$

$$-0.5x = \ln\left(\frac{100-y}{99y}\right)$$

$$P^{-1}(y) = x = \frac{\ln\left(\frac{100-y}{99y}\right)}{-0.5}$$

③ switch variables

$$P^{-1}(x) = \frac{\ln\left(\frac{100-x}{99x}\right)}{-0.5}$$